CERTAIN MINIMAX CONTROL PROBLEMS WITH INCOMPLETE INFORMATION

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A. A. MELIKIAN and F. L. CHERNOUS'KO
(Moscow)
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We apply the minimax (the guarantee) approach to the optimal control problems for plants with incomplete information. We assume that the location of the phase vector is known to within a certain set to which it belongs; the system's position is updated by measurements during the motion. Using the dynamic programing method we investigate a way of contructing the control as a function of measurement results and of time. Similar problems were analyzed in [1 - 3].

1. Discrete case. A discrete system is described by the equations

$$x_{k+1} = F(x_k, u_k, t_k) \qquad (k = 0, 1, 2, \ldots, N), \qquad x_k, F \in E^n \tag{1.1}$$
$$t_0 < t_1 < \ldots < t_N < t_{N+1} = T$$

Here the controls u_k are m-dimensional vectors, $u_k \in E^m$, or are m-dimensional vector-valued functions defined on the interval $[t_k, t_{k+1})$. The components of vector F are, respectively, functions (or functionals) depending on u_k. Furthermore, $u_k \in U_k \subset E^m$, where U_k is a closed bounded set. On the trajectories of system (1.1) we define the functional

$$J = \sum_{k=0}^{N} R_k(u_k) + R_{N+1}(x_{N+1}) \tag{1.2}$$

Here R_k $(k = 0, 1, \ldots, N + 1)$ are continuous functions (or functionals) of their arguments.

At the initial instant t_0 the vector x_0 is specified imprecisely; we know only that $x_0 \in B_0$, where B_0 is a closed set, $B_0 \subset E^n$. Having substituted certain controls u_k $(k = 0, 1, \ldots, N)$ into (1.1), we can obtain a set of possible positions of the phase vector at the instant t_k $(k = 1, 2, \ldots, N + 1)$. We assume that the set of possible positions is updated at each step by means of measurements. Let us describe these measurements. Suppose that the set B_k of possible positions is known at an instant t_k, i.e., the set such that $x_k \in B_k$. It is obvious that at the next instant $x_{k+1} \in F(B_k, u_k, t_k)$. We have here adopted the notation

$$F(B_k, u_k, t_k) = \{x \in E^n : x = F(y, u_k, t_k),\ y \in B_k\}.$$

Let us assume that measurement made at the instant t_{k+1} yields a certain set D_{k+1} such that $x_{k+1} \in D_{k+1}$. Therefore, the vector x_{k+1} belongs also to the intersection $D_{k+1} \cap F(B_k, u_k, t_k) = B_{k+1}$. The measurement results should satisfy the following consistency condition: the intersection B_{k+1} $(k = 0, 1, \ldots, N)$ is not empty. By definition, $D_0 = B_0$. For each instant we can define a certain class of sets $\{D_k\}$ whose elements may appear at D_k (the class measurement outcomes). We take it that the class $\{D_k\}$ consists of all sets obtained by means of shifts in the space E^n from some arbitrary closed convex set. For example, we can take the class $\{D_k\}$ as consisting

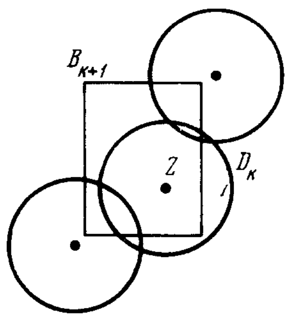

Fig. 1.

of n-dimensional spheres with center at some point $z \in E^n$ and with a specified diameter $d_k > 0$; the choice of the vector z also determines the actual measurement outcome D_k (Fig. 1).

We assume that the controls u_k are chosen after the set D_k (respectively, also B_k) have become known. We pose the following problem: knowing the measurement possibilities, i.e., the classes $\{D_k\}$ $(k=1, 2, \ldots, N+1)$, and the initial set B_0, find admissible controls $u_k(B_k)$ $(k=0, 1, \ldots, N)$ as functions of the measurement results, which ensure the minimal guaranteed value of functional (1.2) on the trajectories of system (1.1). Let us extend the problem posed and seek controls furnishing a minimal guaranteed value to the functional

$$J_i = \sum_{k=i}^{N} R_k(u_k) + R_{N+1}(x_{N+1}) \qquad (i=0,1,\ldots,N) \tag{1.3}$$

for a fixed i, on the trajectories of system (1.1) for $i \leqslant k \leqslant N$. The set B_i of possible positions $(x_i \in B_i)$ is given at the instant t_i.

We define the Bellman function by the formula

$$S(B_i, t_i) = \min\nolimits_{u_i} \max\nolimits_{D_{i+1}} \min\nolimits_{u_{i+1}} \max\nolimits_{D_{i+2}} \cdots$$
$$\cdots \min\nolimits_{u_N} \max\nolimits_{D_{N+1}} \max\nolimits_{x_{N+1}} J_i \qquad (i=0,1,\ldots,N) \tag{1.4}$$

The function $S(B_i, t_i)$ equals the minimal value which can be guaranteed for functional J_i if at the instant t_i we have the set B_i. We assume that all the extrema in (1.4) are reachable. The operation distribution order in (1.4) is chosen in correspondence with the order of information inflow and of control information. The minimum with respect to u_k ranges over the set U_k. The rightmost maximum in (1.4) is computed with respect to the set $B_{N+1} = D_{N+1} \cap F(B_N, u_N, t_N)$, the remaining maxima range over $D_k \in \{D_k\}$, satisfying the consistency condition. Later on we shall need one property of the function $S(B_i, t_i)$, which is easily verifiable with the aid of (1.4),

$$S(B_i, t_i) \leqslant S(B_i', t_i), \quad \text{if} \quad B_i \subseteq B_i' \tag{1.5}$$

We transform (1.4), with due regard to (1.3),

$$S(B_i, t_i) = \min\nolimits_{u_i} \max\nolimits_{D_{i+1}} \Big[R_i(u_i) + \min\nolimits_{u_{i+1}} \max\nolimits_{D_{i+2}} \cdots$$
$$\ldots, \min\nolimits_{u_N} \max\nolimits_{D_{N+1}} \max\nolimits_{x_{N+1}} \Big(\sum_{k=i+1}^{N} R_k(u_k) + R_{N+1}(x_{N+1}) \Big) \Big]$$

Note that the second term within the brackets, in accordance with (1.4), equals $S(B_{i+1}, t_{i+1})$, where $B_{i+1} = D_{i+1} \cap F(B_i, u_i, t_i)$. Consequently, the function S satisfies the recurrence relation

$$S(B_i, t_i) = \min\nolimits_{u_i} \max\nolimits_{D_{i+1}} [S(D_{i+1} \cap F(B_i, u_i, t_i), t_{i+1}) + R_i(u_i)] \tag{1.6}$$
$$u_i \in U_i, \; D_{i+1} \in \{D_{i+1}\}, \; D_{i+1} \cap F(B_i, u_i, t_i) \neq \phi \qquad (i=0,1,\ldots,N)$$

We further set

$$S(B_{N+1}, T) = \max_{x_{N+1} \in B_{N+1}} R_{N+1}(x_{N+1})$$

Knowing the function $S(B_i, t_i)$ for all regions B_i capable of being realized at the instant t_i $(i = 0, 1, \ldots, N)$, we can obtain the control $u_i(B_i)$ by carrying out the operations in (1.6). This problem is highly complicated for the case of arbitrary regions. We make some simplifying assumptions:

1). Equations (1.1) are linear in $x_k : F(x_k, u_k, t_k) = A_k x_k + b_k(u_k)$, where A_k is a square $n \times n$ matrix, $b_k \in E^n$.

2). The set B_0 is a segment in E^n. We denote this segment $I(x^1, x^2)$, where x^1, $x^2 \in E^n$ are the endpoints of the segment.

From these assumptions it follows that all regions B_i $(i = 1, 2, \ldots, N + 1)$ are segments, $B_i = I(x_i^1, x_i^2)$. Thus the structure of sets B_i is simplified and they are specified by $2n$-parameters (the coordinates of the vectors x_i^1, x_i^2). We can now treat the function S as a function of $2n + 1$ variables, $S(x_i^1, x_i^2; t_i)$. It is evident that $S(x_i^1, x_i^2, t_i) = S(x_i^2, x_i^1, t_i)$. Note that assumption (2) signifies that the initial error with respect to one linear combination of phase coordinates is most significant.

Assumptions (1) and (2) allow us to simplify (1.6) essentially. On the unit sphere we define the function

$$d(e, t_{i+1}) = \max_{D_{i+1} \in \{D_{i+1}\}} |x_{i+1}^2 - x_{i+1}^1| \quad (i = 0, 1, \ldots, N) \qquad e \in E^n, \ |e| = 1$$

The absolute value sign denotes the length of the vectors in E^n. The points x_{i+1}^1, x_{i+1}^2 are the points of intersection of an arbitrary straight line directed along the basis vector e with the boundary of set D_{i+1} (Fig. 2). Let us assume, for example, that the scalar product $w_k = (c_k, x_k) + \Delta_k$, is measured, where the error $|\Delta_k| \leqslant \nu_k$, and c_k and ν_k are given quantities, $\nu_k \geqslant 0$. Then, the sets $D_k = \{x \in E^n : w_k - |\Delta_k| \leqslant (c_k, x) \leqslant w_k + |\Delta_k|\}$ are the sets $D_k \in \{D_k\}$. The function $d(e, t_k)$ is easily computed in this case and equals $2\nu_k \, |(c_k, e)|^{-1}$. The function $d(e, t_k)$ introduced characterizes the measurement possibilities.

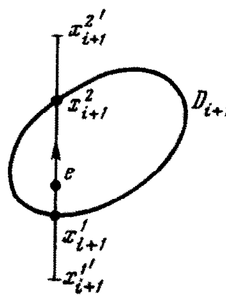

Fig. 2.

Suppose that the segment $I(x_i^1, x_i^2)$ is known at the instant t_i. At the next instant t_{i+1}, in accordance with assumption (1), it is once again mapped onto a certain segment $I(x_{i+1}^{1'}, x_{i+2}^{2'})$. If at the ith step it turns out that $|x_{i+1}^{2'} - x_{i+1}^{1'}| \leqslant d(e_{i+1}, t_{i+1})$, where $e_{i+1} = (x_{i+1}^{2'} - x_{i+1}^{1'})\,|x_{i+1}^{2'} - x_{i+1}^{1'}|^{-1}$, then there exists a set $D_{i+1}' \in \{D_{i+1}\}$ such that $I(x_{i+1}^{1'}, x_{i+1}^{2'}) \subseteq D_{i+1}'$. Recalling property (1.5) of the function S, we can note that the maximum in (1.6) is reached, for example, on this set D_{i+1}'. In this case, in (1.6) we can omit the maximization with respect to D_{i+1} and rewrite this relation as

$$S(x_i^1, x_i^2, t_i) = \min_{u_i \in U_i} [S(x_{i+1}^{1'}, x_{i+1}^{2'}, t_{i+1}) + R_i(u_i)] \tag{1.7}$$

This case signifies that the measurement is ineffective at the instant t_{i+1} i.e., does not guarantee an increase in information on the system's position.

We consider the other case, when $|x^{2'}_{i+1} - x^{1'}_{i+1}| > d(e_{i+1}, t_{i+1})$ (the measurement is effective). Let x^1_{i+1}, x^2_{i+1} be the points of intersection of the segment $I(x^{1'}_{i+1}, x^{2'}_{i+1})$ with the set D_{i+1}. From the same property (1.5) of function S it follows that the maximization in (1.6) need be carried out only with respect to those $D_{i+1} \in \{D_{i+1}\}$ for which $|x^2_{i+1} - x^1_{i+1}| = d(e_{i+1}, t_{i+1}), x^1_{i+1}, x^2_{i+1} \in I(x^{1'}_{i+1}, x^{2'}_{i+1})$. For such sets D_{i+1} the vectors x^1_{i+1}, x^2_{i+1} can be represented as (Fig. 2)

$$x^1_{i+1} = x^{1'}_{i+1} + \alpha(x^{2'}_{i+1} - x^{1'}_{i+1}), \qquad x^2_{i+1} = x^{1'}_{i+1} + \alpha'(x^{2'}_{i+1} - x^{1'}_{i+1}) \tag{1.8}$$

$$\alpha' = \alpha + d(e_{i+1}, t_{i+1})\,|x^{2'}_{i+1} - x^{1'}_{i+1}|^{-1}, \qquad e_{i+1} = (x^{2'}_{i+1} - x^{1'}_{i+1})\,|x^{2'}_{i+1} - x^{1'}_{i+1}|^{-1}$$

where α is a real parameter. From the condition $I(x^1_{i+1}, x^2_{i+1}) \subseteq I(x^{1'}_{i+1}, x^{2'}_{i+1})$ we obtain that α should lie within the limits

$$0 \leqslant \alpha \leqslant 1 - d(e_{i+1}, t_{i+1})\,|x^{2'}_{i+1} - x^{1'}_{i+1}|^{-1} \tag{1.9}$$

As follows from the description of the classes $\{D_k\}$ a possible measurement outcome corresponds to each α from the interval indicated. Thus, a one-parameter family has been picked out from the worst measurement outcomes, and the maximum with respect to D_{i+1} in (1.6) can be replaced by a maximum with respect to α. Indeed, having noted that in the case being considered

$$S(D_{i+1} \cap F(B_i, u_i, t_i)\, t_{i+1}) = S(x^1_{i+1}, x^2_{i+1}, t_{i+1})$$

we can write (1.6) in the form

$$S(x_i^1, x_i^2, t_i) = \min_{u_i \in U_i} \max_\alpha [S(x^1_{i+1}, x^2_{i+1}, t_{i+1}) + R_i(u_i)] \tag{1.10}$$

where α varies in the interval (1.9) and the vectors x^1_{i+1}, x^2_{i+1} are chosen in form (1.8). Equation (1.10) simplifies under the change of variables

$$z_i = {}^1/_2(x_i^1 + x_i^2), \qquad y_i = x_i^2 - x_i^1 \tag{1.11}$$

Now a measurement outcome yields a pair of vectors z_i, y_i. The phase vector x_i can take the values $x_i = z_i + \beta y_i$, $|\beta| \leqslant {}^1/_2$. If the vectors z_i, y_i are known at instant t_i then, as above, the primes distinguish the vectors into which z_i, y_i, are taken in accordance with (1.11) and assumption (1),

$$z'_{i+1} = A_i z_i + b_i(u_i), \quad y'_{i+1} = A_i y_i \quad (i = 0, 1 \ldots, N)$$

$$e_{i+1} = y'_{i+1}\,|y'_{i+1}|^{-1} = A_i y_i\,|A_i y_i|^{-1} \tag{1.12}$$

The last relation in (1.12) follows from (1.8), (1.11). The vectors z_{i+1}, y_{i+1} are found as a result of measurement, i.e., by the intersection of the segment $I(z'_{i+1} - {}^1/_2 y'_{i+1}, z'_{i+1} + {}^1/_2 y'_{i+1})$ with some set $D_{i+1} \in \{D_{i+1}\}$. We take it that the Bellman function depends upon the arguments z_i, y_i, t_i, and for it we retain the previous notation $S(z_i, y_i, t_i)$. It is not difficult to see (relation (1.7)) that for the instants t_i for which $|y'_{i+1}| \leqslant d(e_{i+1}, t_{i+1})$ we have

$$S(z_i, y_i, t_i) = \min_{u_i \in U_i} [S(A_i z_i + b_i(u_i), A_i y_i, t_{i+1}) + R_i(u_i)] \tag{1.13}$$

If the inequality $|y'_{i+1}| > d(e_{i+1}, t_{i+1})$ is fulfilled, then analogously to what we did before (relation (1.10)), we pick out a one-parameter family of the worst measurement outcomes. From (1.8), (1.12) we have

$$y_{i+1} = x^2_{i+1} - x^1_{i+1} = d(e_{i+1}, t_{i+1})\, y'_{i+1}\,|y'_{i+1}|^{-1} = d(e_{i+1}, t_{i+1})\, A_i y_i\,|A_i y_i|^{-1} \tag{1.14}$$

i.e., the vector y_{i+1} is one and the same for the various measurement outcomes. From (1.8), in accordance with (1.11), (1.12), for the vector z_i we obtain

$$z_{i+1} = z'_{i+1} + \beta y'_{i+1} = A_i z_i + b_i(u_i) + \beta A_i y_i \tag{1.15}$$

where the parameter β takes the values (see (1.9))

$$|\beta| \leqslant {}^1/_2 (1 - d(e_{i+1}, t_{i+1})\,|A_i y_i|^{-1}), \qquad \beta = \alpha - {}^1/_2 (1 - d(e_{i+1}, t_{i+1})\,|A_i y_i|^{-1} \tag{1.16}$$

In the new variables (1.10) has the form

$$S(z_i, y_i, t_i) = \min_{u_i \in U_i} \max\nolimits_\beta [S(z_{i+1}, y_{i+1}, t_{i+1}) + R_i(u_i)] \tag{1.17}$$

where z_{i+1}, y_{i+1} are taken from (1.14) and (1.15) respectively, and the parameter β varies within the limits (1.16). For $t = t_{N+1} = T$ the function S is determined by the relation

$$S(z_{N+1}, y_{N+1}, T) = \max\nolimits_\beta R_{N+1}(x_{N+1}), \quad x_{N+1} = z_{N+1} + \beta y_{N+1}, |\beta| \leqslant {}^1/_2 \tag{1.18}$$

By using the initial condition (1.18) and the recurrence relations (1.17) or (1.13), depending on whether or not the measurement is effective (i.e., on whether or not the inequality $|y'_{i+1}| > d(e_{i+1}, t_{i+1})$ is fulfilled), we can compute the function $S(z_i, y_i, t_i)$ successively for the instants t_i $(i = N, N-1, \ldots, 1, 0)$ and obtain, in passing, the values of the control guaranteeing the value (1.4) to functional (1.3).

2. Continuous system, discrete observations. We are given the system

$$x^\cdot = A(t)x + b(u, t), \quad t \in [t_0, T], \quad b, x \in E^n, \quad u(t) \in U_t \subset E^n \tag{2.1}$$

Here U_t is a compactum for all $t \in [t_0, T]$, $A(t)$ is an $n \times n$ matrix depending on t. On the trajectories of system (2.1) the following functional is defined:

$$J = \int_{t_0}^{T} f_0(u, t)\,dt + R_T(x_T), \quad x_T = x(T) \tag{2.2}$$

Here, f_0 and R_T are continuous functions. The set B_0 of possible initial states $x(t_0) \in \in B_0$, which, in particular, may be a segment, is known. At specified instants $t_0 < < t_1 < \ldots < t_N < t_{N+1} = T$ the system's state is updated by means of measurements. Analogous to what was presented in Sect. 1, we can describe the classes $\{D_k\}$. The control on the interval $[t_k, t_{k+1})$ is chosen after the measurements at the instant t_k. The problem of obtaining a minimal guaranteed value of functional (2.2) reduces to the problem in Sect. 1.

Let us show this. Let $X(t, \tau)$ be the Cauchy matrix of the system $x^\cdot = A(t)x$ [4]. We replace system (2.1) by the equivalent discrete system

$$x_{k+1} = X(t_{k+1}, t_k)\,x_k + \int_{t_k}^{t_{k+1}} X(t_{k+1}, \tau)\,b(u_k(\tau), \tau)\,d\tau \tag{2.3}$$

$$x_0 \in B_0, \qquad x_k = x(t_k) \qquad (k = 0, 1, \ldots, N)$$

Here $u_k(t)$ are functions given on the interval $[t_k, t_{k+1})$ with values in U_t. Functional (2.2) is reduced to form (1.2) by the substitution

$$R_k(u_k) = \int_{t_k}^{t_{k+1}} f_0(u_k(\tau), \tau)\,d\tau, \; R_{N+1} = R_T$$

The Bellman function is defined analogously. It satisfies relations (1.4), (1.6) or (1.13), (1.17), (1.18).

3. Examples. 1. Consider the scalar system

$$x_{k+1}=x_k+b_k u_k,\ |u_k|\leqslant\mu,\ b_k>0\quad (k=0,1,\ldots,N),\quad \mu>0 \tag{3.1}$$

At the initial instant the value x_0 is given with the error $a_1\leqslant x_0\leqslant a_2$. The measurement function $d(e,t_k)=d_k$, $d_k>0$ $(k=0,1,\ldots,N+1)$ is known and, moreover, $d_0=a_2-a_1$. The functional to be minimized has the form $J=R_{N+1}(x_{N+1})=|x_{N+1}|$. The following problems, for example, are continuous analogs of (3.1):

$$1)\ x_1^{\cdot}=b(t)\,u,\qquad 2)\ x_1^{\cdot}=x_2,\qquad x_2^{\cdot}=u$$

$$|u|\leqslant\mu,\qquad t_0\leqslant t\leqslant T,\qquad J=|x_1(T)|-\min$$

where the coordinate x_1 is measured. System (2) reduces to form (1) by the substitution $\xi(t)=x_1(t)+(T-t)\,x_2(t)$; here the functional to be minimized still has the same form because $\xi(T)=x_1(T)$.

Returning to system (3.1), we can at once write down the solution of Eq. (1.13): $y_k=d_k$ $(k=0,1,\ldots,N+1)$ under the assumption that $|y_{k+1}|>d_{k+1}$, i.e. $\Delta d_k=d_{k+1}-d_k<0$ $(k=0,1,\ldots,N)$. This assumption signifies that the measurement accuracy increases with time, and all measurements are effective. Equation (1.15), giving the motion of the center of the segment of indeterminacy, is written, after the change $\beta_k=(\Delta d_k/d_k)\,v_k$ in the form

$$z_{k+1}=z_k+b_k u_k+\Delta d_k v_k,\ |u_k|\leqslant\mu,\ |v_k|\leqslant 1/2\ (k=0,1,\ldots,N)$$

For the Bellman function we have, in accordance with (1.18), the initial condition

$$S(z_{N+1},t_{N+1})=|z_{N+1}|+1/2\,d_{N+1}$$

With the aid of (1.17) we compute the value of the Bellman function at the instant t_N

$$S(z_N,t_N)=\min_{u_N}\max_{v_N}|z_N+b_N u_N+\Delta d_N v_N|+1/2\,d_{N+1}$$

$$|u_N|\leqslant\mu,\ |v_N|\leqslant 1/2 \tag{3.2}$$

To be specific let us assume that the quantity $l_i=b_i\mu+1/2\,\Delta d_i$ decreases as i grows, and moreover $l_i>0$ for $i\leqslant m$ and $l_i<0$ for $i>m$, where $0<m<N$. The computation of the maximum in (3.2) yields the following value v_N:

$$v_N=-1/2\operatorname{sgn} z_N,\ |z_N|\neq 0;\ v_N=\pm 1/2,\ |z_N|=0 \tag{3.3}$$

For what is to follow we remark that analogous formulas for the "control" v_k are obtained also at all the other instants t_k.

The Bellman function and the control u_N at the instant t_N are

$$S(z_N,t_N)=|z_N|-b_N\mu+1/2\,d_N,\qquad u_N=2\mu\,v_N$$

Using (1.16) repeatedly, we can obtain

$$S(z_k,t_k)=|z_k|+1/2\,d_k-\mu\sum_{i=k}^{N}b_i,\ u_k=2\mu v_k\ (k=m+1,m+2,\ldots,N)$$

For the instant t_m, where $l_m>0$ the following values are obtained:

$$S(z_m,t_m)=|z_m|+1/2\,d_m-\mu\sum_{i=m}^{N}b_i,\ u_m=2\mu v_m\qquad \text{for}\ |z_m|\geqslant l_m$$

$$S(z_m, t_m) = {}^1/_2\, d_{m+1} - \mu \sum_{i=m+1}^{N} b_i,\ u_m = 2v_m b_m^{-1}(|z_m| - {}^1/_2\, \Delta d_m) \quad \text{for} \quad |z_m| < l_m$$

At the remaining instants t_k $(k \leqslant m-1)$ we obtain

$$u_k = 2\mu v_k,\ S(z_k, t_k) = \begin{cases} |z_k| + {}^1/_2\, d_k - \mu \sum\limits_{i=k}^{N} b_i & \text{for} \quad |z_k| \geqslant \sum\limits_{i=k}^{m} l_i \\ {}^1/_2\, d_{m+1} - \mu \sum\limits_{i=m+1}^{N} b_i & \text{for} \quad z_k| < \sum\limits_{i=k}^{m} l_i \end{cases} \tag{3.4}$$

In the formulas presented, v_k is determined by formulas (3.3) with the subscript N replaced by k. In case $|z_k| < \sum\limits_{i=k}^{m} l_i$ the control u_k is determined ambiguously; only one of the possible values has been presented in (3.4). As we see, in the region between the polygonal line $|z_k| = l_k + \ldots + l_m$, $k \leqslant m$ and the interval $[t_0, t_{m+1}]$ the Bellman function is constant and is given by the lower formula in (3.4). In the general case when the quantity l_i changes sign several times, there may exist several such regions of constancy of the function, with their own values of the constant.

2. Let the system be described, as before, by Eq. (3.1) with the same measurements, the same informational conditions, and the same functional to be minimized as in Example 1. We impose an integral constraint on the control resource

$$\sum_{k=0}^{N} |u_k| \leqslant q_0, \qquad q_0 > 0$$

In order to apply the proposed procedure to this problem we introduce an additional phase coordinate q_k, subjecting it to the equation

$$q_{k+1} = q_k - |u_k| \qquad (k = 0, 1, \ldots, N) \tag{3.5}$$

We take it also that $|u_k| \leqslant q_k$ or $q_{k+1} \geqslant 0$ $(k = 0, 1, \ldots, N)$. Then, from (3.5) it follows that

$$\sum_{i=k}^{N} |u_i| \leqslant q_k$$

i.e., q_k is the control resource remaining. The equation for the coordinate z_k has the previous form and together with (3.5) gives a second-order system

$$z_{k+1} = z_k + b_k u_k + \Delta d_k v_k, \qquad |u_k| \leqslant q_k, \qquad |v_k| \leqslant {}^1/_2$$
$$q_{k+1} = q_k - |u_k| \qquad (k = 0, 1, \ldots, N) \tag{3.6}$$

The dependency of the sets U_k $(u_k \in U_k)$ on the phase coordinates was not assumed in the derivation of (1.17). However, it is not difficult to obtain, by tracing the derivation of (1.17), that the Bellman function $S(z_k, q_k, t_k)$ for problem (3.6) satisfies an analogous equation. The latter is occasioned by the fact that the coordinate q_k defining the constraint on the control is known (can be measured) precisely. We derive, omitting details, the solution of this equation under the additional condition $b_{k+1} \leqslant b_k$ $(k = 0, 1, \ldots, N-1)$ signifying that the effectiveness of the control does not increase with time. This solution has the form

$$S(z_k, q_k, t_k) = \max\{0, f_{kk}, f_{k\,k+1}, \ldots, f_{kN}\} \ (k = 0, 1, \ldots, N) \tag{3.7}$$

$$f_{kn} = \frac{b_n}{b_k}(|z_k| - b_k q_k - g_{kn}),\ g_{kn} = \frac{b_k}{2} \sum_{i=k}^{n} \frac{\Delta d_i}{b_i} + \frac{b_k}{2b_n} \sum_{i=n+1}^{N} \Delta d_i \ (n = k, k+1, \ldots, N)$$

Solution (3.7) can be further written in the following form:

$$S(z_k, q_k, t_k) = \begin{cases} 0 & \text{for } \xi_k \leqslant p_k = \frac{b_k}{2} \sum_{i=k}^{N} \frac{\Delta d_i}{b_i} \\ f_{kn} & \text{for } p_{n-1} < \xi_k \leqslant p_n \, (n = k+1, k+2, \ldots, N) \\ f_{kk} & \text{for } \xi_k > p_N = {}^1/_2 \Delta d_k \end{cases}$$

$$\xi_k = |z_k| - b_k q_k, \quad p_n = (b_n - b_{n+1})^{-1} (b_n g_{kn} - b_{n+1} g_{kn+1}) \, (n = k+1, \ldots, N-1)$$

The values of u_k, v_k supplying the minimax are given by the relations

$$v_k = -{}^1/_2 \operatorname{sgn} z_k, \; |z_k| \neq 0; \qquad v_k = \pm {}^1/_2, \; |z_k| = 0$$

$$u_k = 2 v_k q_k, \quad \text{if} \quad \xi_k > {}^1/_2 \Delta d_k \tag{3.8}$$

$$u_k = u_k^* \equiv 2 v_k b_k^{-1} (|z_k| - {}^1/_2 \Delta d_k), \quad \text{if} \quad p_k \leqslant \xi_k \leqslant {}^1/_2 \Delta d_k \; (k = 0, 1, \ldots, N)$$

For $\xi_k < p_k$ the optimal control is determined ambiguously; for example, we can take $u_k = u_k^*$. The synthesis (3.8) drives the center of the indeterminacy segment to zero at the initial step and keeps it equal to zero until the control resource is exhausted. If, however, the resource does not permit z_k to vanish, then the whole supply is consumed at the very first step. We note two special cases of (3.7). If the condition $b_k \equiv b$ ($k = 0, 1, \ldots, N$), is fulfilled, then $f_{kk} = f_{k\,k+1} = \ldots = f_{kN} = |z_k| - bq_k - {}^1/_2 (d_{N+1} - d_k)$, which simplifies the solution. However, under the assumption that the measurements are precise, i.e., $d_k = 0$ ($k = 0, 1, \ldots, N+1$), from (3.7), (3.8) we obtain the obvious unambiguous solution of the problem

$$S(z_k, q_k, t_k) = \max\{0, f_{kk}, f_{k+1}, \ldots, f_{kN}\} = \max\{0, f_{kk}\} = \max\{0, |z_k| - b_k q_k\}$$

$$u_k = -\operatorname{sgn} z_k q_k, \qquad \text{if} \quad \xi_k > 0; \qquad u_k = -z_k b_k^{-1}, \quad \text{if} \quad \xi_k \leqslant 0$$

4. Continuous system, continuous observations. We can show that a formal passing to the limit in the relations in Sect. 1 as $\Delta t = \max_k \{t_{k+1} - t_k\} \to 0$ permits us to obtain corresponding equations for the case of a continuous observation. Let us take the interval between observations to be equal and let us set $t_{i+1} - t_i = \Delta t = (T - t)(N+1)^{-1}$. We replace the index i by the argument $t = t_0 + i\Delta t$, and the index $i+1$ by the argument $t + \Delta t$. Passing to the limit as $\Delta t \to 0$ means that the observation instants are made more frequent. In the limit the observation is continuous. Consequently, the classes $\{D_t\}$ should be described for each instant $t \in [t_0, T]$.

We write Eq. (2.1) in the form

$$x(t + \Delta t) = x(t) + \Delta t A(t) x(t) + \Delta t b(u(t), t) + o(\Delta t^2) \tag{4.1}$$

In the relations of Sect. 1 we should replace the matrix A_i by $E + \Delta t A(t)$, where E is the unit matrix. For simplicity we take it that for any Δt the inequality

$$|y'_{i+1}| = |A_i y_i| > d(e_{i+1}, t_{i+1}) \tag{4.2}$$

is satisfied at each step, signifying, according to Sect. 1, that the measurements are always effective. Then, from relations (1.12), (1.14) follows $e_{i+1} = y_{i+1} |y_{i+1}|^{-1}$ for all i. By replacing indices by arguments in the equality just obtained and in the last equality of (1.12), we obtain $e(t) = y|y|^{-1}$, $e(t + \Delta t) = (y + \Delta t A y)|y + \Delta t A y|^{-1}$, where the argument t has been dropped in the right-hand sides. By passing to the limit as $\Delta t \to 0$, we have

$$e^{\cdot} = P(e, t), \; P(e, t) = A(t)e - (A(t)e, e)e, \; t \in [t_0, T], \; e(t_0) = y_0 \,|\, y_0 \,|^{-1} \tag{4.3}$$

We can remark that system (4.3) describes the variation of the unit vector parallel to the solution vector of the system $y^{\cdot} = Ay$, $y(t_0) = y_0$.

To obtain a continuous analog of condition (4.2) let us ascertain to what the ratio $d(e_{i+1}, t_{i+1}) \,|\, A_i y_i \,|^{-1}$ tends as $\Delta t \to 0$. Note that after the indices have been replaced by arguments, from (1.14) follows $d(e(t), t) = |\, y(t) \,|$ for all t. Further, with due regard to (4.3), we have

$$\frac{d(e(t+\Delta t), t+\Delta t)}{|\, y(t) + \Delta t A(t) y(t) \,|} = 1 - \frac{1}{|\, y \,|} D(e, t)\Delta t + o(\Delta t^2) \tag{4.4}$$

$$D(e, t) = d(e, t)(A(t)e, e) - \frac{\partial d}{\partial t} - \left(P(e, t), \frac{\partial d}{\partial e}\right)$$

By $\partial d / \partial e$ we have denoted the gradient of the scalar function $d(e, t)$ along the vector argument e. In the limit inequality (4.2) becomes the inequality

$$D(e, t) > 0 \tag{4.5}$$

In (1.14), by passing to the continuous argument t and by letting $\Delta t \to 0$ with due regard to (4.4) we obtain

$$y^{\cdot} = Ay - De, \quad y(t_0) = y_0, \quad e = y \,|\, y \,|^{-1} \tag{4.6}$$

From (1.14) it follows also that the solution of Eq. (4.6) has the form $y(t) = d(e(t), t)\, \xi(t) \,|\, \xi(t) \,|^{-1}$, where the vector $\xi(t)$ is the solution of the homogeneous equation $\xi^{\cdot} = A(t)\xi$, $\xi(t_0) = y_0$. This fact is easily verified also by a direct substitution into system (4.6). Consequently, the solution of Eq. (4.3) has the form $e(t) = \xi(t) \,|\xi(t) \,|^{-1}$.

The interval (1.16) of variation of β shrinks to a point as $\Delta t \to 0$ (see (4.4)). Therefore we introduce another parameter p by the relation $\beta = p\Delta t$. The parameter β is determined by the position of the segment $I(x_k^1, x_k^2)$ on the segment $I(x_k^{1'}, x_k^{2'})$ (Fig. 2), while the parameter p corresponds to the velocity of the motion of one segment relative to the other (the variation rate of the measurement result). Having divided both sides in (1.16) by Δt and passing to the limit as $\Delta t \to 0$, analogously to (4.4) we obtain the interval of variation of parameter p

$$|\, p \,| \leqslant {}^1/_2 D(e, t) \,/\, |\, y \,| \tag{4.7}$$

After the change of variables $p = vD \,|\, y \,|^{-1}$ Eq. (1.14) becomes the equation

$$z^{\cdot} = Az + b(u, t) + vD(e, t)\, e, \; z(t_0) = z_0, \; u(t) \in U_t, \; |\, v(t) \,| \leqslant {}^1/_2 \tag{4.8}$$

Equations (4.3) and (4.6) are solved independently of v and u, therefore, we can reckon that (4.8) yields a differential game with the phase vector z, where the controls of the antagonists are the vector u and the scalar v, and the payoff is the functional (2.2). Knowing the vectors $z(t)$ and $y(t)$, we can assert that $x(t) = z(t) + \gamma y(t)$, where $|\, \gamma \,| \leqslant {}^1/_2$, $t \in [t_0, T]$. Thus, the problem of controlling a continuous system under an imprecise knowledge of the phase vector is reduced in the case being considered to the differential game (4.8), (2.2) where we can apply the methods for solving differential games [1, 5].

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SUFFICIENT OPTIMALITY CONDITIONS IN DIFFERENTIAL GAMES

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M. BAIBAZAROV
(Sverdlovsk)
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We consider game problems in which the payoff is some function of the terminal state of a conflict-controlled system. We state sufficient conditions for the existence of optimal minimax and maximin strategies of the players. We show that optimal strategies exist if the corresponding Bellman equation has a solution. We consider the question of the existence of optimal strategies both in the class of deterministic as well as in the class of mixed strategies. The reasoning presented is based on the results in [1, 2]. The questions considered border on the investigations presented in [2 - 5].

1. Let the motion of a conflict-controlled system be described by the nonlinear equation

$$dx/dt = f(t, x, u, v) \tag{1.1}$$

Here x is the n-dimensional phase vector, u and v are the controls of the first and second players, respectively, $f(t, x, u, v)$ is a continuous vector-valued function satisfying a Lipschitz condition in x. The realizations $u[t]$ and $v[t]$ of the controls u, and v are constrained by the conditions $u[t] \in P(t)$ and $v[t] \in Q(t)$, where $P(t)$ and $Q(t)$ are closed, bounded and convex sets in the corresponding vector spaces, varying continuously with t. We assume that the right-hand side of system (1.1) satisfies the condition

$$|x'f(t, x, u, v)| \leqslant \lambda(1 + \|x\|^2), \quad u \in P(t), \quad v \in Q(t), \quad t \in [t_0, \vartheta]$$

The payoff is the quantity $w(x[\vartheta])$, defined at the final instant $t = \vartheta$ by the position $x[\vartheta]$ realized. The function $w(x)$ is assumed continuous. Thus, we are considering a game with a fixed final instant $t = \vartheta$. The first player strives to minimize the quantity $w(x[\vartheta])$ under the most adverse behavior of the second player. The second player's problem is to ensure a completion of the game with the largest possible value of the payoff.

We emphasize that the controls u and v should be formed by a feedback rule in order